

Weakly symmetry of a class of g -natural metrics on tangent bundles

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Abstract

Considering the class G of g -natural metrics on the tangent bundle of a Riemannian manifold (M, g) , it is shown that the flatness for g is a necessary and sufficient condition of weakly symmetry (recurrent or pseudo-symmetry) of G . In particular, the cases of weakly symmetric Sasakian lift metric studied by Bejan and Crasmareanu and recurrent or pseudo-symmetric Sasakian lift metric studied by Binh and Tamásy are obtained.

Keywords: g -natural metric, Weakly symmetric Riemannian manifold.

1 Introduction

In [8], Tamásy and Binh introduced the notation of weakly symmetric Riemannian manifold which is a stronger variant of recurrent and pseudo-symmetric manifolds. Then they studied the weak symmetries of Einstein and Sasakian manifolds in [9]. Recent studies show that the notion of weakly symmetry has an important role in Riemannian geometry [2]-[10].

In [2], Bejan and Crasmareanu considered the Sasakian lift g^s to the tangent bundle of a Riemannian manifold (M, g) and proved that the weakly symmetry of g^s is equivalent to the flatness for g and g^s . Indeed, they extended the result obtained by Tamásy and Binh [3] for recurrent and pseudo-symmetric manifolds. Moreover, in [2] the authors provided the following open problem: to extend the present result to other classes of metrics on tangent bundles. To solving of this open problem, we consider the metric $G = ag^s + bg^h + cg^v$ (a, b, c are constants) which is a class of g -natural metrics introduced by Abbassi and Sarik in [1] and we show that (TM, G) is weakly symmetric (recurrent or pseudo-symmetric) Riemannian manifold if and only if (M, g) is flat.

2 Preliminaries

Let (M, g) be a Riemannian manifold with dimension $n \geq 3$ and TM its tangent bundle. If we consider coordinate system $x = (x^i)$ on the base manifold M and

corresponding coordinates $(x, y) = (x^i, y^i)$ on TM , then the metric g has the local coefficients $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. Let ∇ be a Riemannian connection on M with coefficients Γ_{ij}^k where $1 \leq i, j, k \leq n$. The Riemannian curvature tensor is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad \forall X, Y, Z \in \mathcal{X}(M).$$

Let π the natural projection from TM to M . Consider $\pi_* : TTM \mapsto TM$ and put

$$\ker \pi_{*v} = \{z \in TTM \mid \pi_{*v}(z) = 0\}, \quad \forall v \in TM.$$

Then the vertical vector bundle on M is defined by $VTM = \bigcup_{v \in TM} \ker \pi_{*v}$. A *horizontal distribution* on TM is a complementary distribution HTM for VTM on TTM . It is clear that HTM is a horizontal vector bundle. By definition, we have the decomposition

$$TTM = VTM \oplus HTM. \quad (2.1)$$

Using the induced coordinates (x^i, y^i) on TM , we can choose a local field of frames $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ adapted to the above decomposition namely $\frac{\delta}{\delta x^i} \in \mathcal{X}(HTM)$ and $\frac{\partial}{\partial y^i} \in \mathcal{X}(VTM)$ are sections of horizontal and vertical sub-bundles HTM and VTM , defined by

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^a \Gamma_{ai}^j \frac{\partial}{\partial y^j}. \quad (2.2)$$

According to (2.1), every vector field \tilde{X} on TM has the decomposition $\tilde{X} = h\tilde{X} + v\tilde{X}$. Moreover, a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M has the vertical lift $X^v = X^i \frac{\partial}{\partial y^i}$ and the horizontal lift $X^h = X^i \frac{\delta}{\delta x^i}$.

2.1 A class of g -natural metrics on tangent bundle

Let g be a Riemannian metric on a manifold M . The Sasaki lift g^s of g is defined by

$$\begin{cases} g_{(x,y)}^s(X^h, Y^h) = g_x(X, Y), & g_{(x,y)}^s(X^h, Y^v) = 0, \\ g_{(x,y)}^s(X^v, Y^h) = 0, & g_{(x,y)}^s(X^v, Y^v) = g_x(X, Y). \end{cases}$$

Also, the horizontal lift g^h and the vertical lift g^v of g are defined as follows [1]

$$\begin{cases} g_{(x,y)}^h(X^h, Y^h) = 0, & g_{(x,y)}^h(X^h, Y^v) = g_x(X, Y), \\ g_{(x,y)}^h(X^v, Y^h) = g_x(X, Y), & g_{(x,y)}^h(X^v, Y^v) = 0, \\ g_{(x,y)}^v(X^h, Y^h) = g_x(X, Y), & g_{(x,y)}^v(X^h, Y^v) = 0, \\ g_{(x,y)}^v(X^v, Y^h) = 0, & g_{(x,y)}^v(X^v, Y^v) = 0. \end{cases}$$

Now we consider the metric $G = ag^s + bg^h + cg^v$, where a, b, c are constants. Indeed we can present G as follows

$$\begin{cases} G_{(x,y)}(X^h, Y^h) = (a+c)g_x(X, Y), & G_{(x,y)}(X^h, Y^v) = bg_x(X, Y), \\ G_{(x,y)}(X^v, Y^h) = bg_x(X, Y), & G_{(x,y)}(X^v, Y^v) = ag_x(X, Y). \end{cases} \quad (2.3)$$

This metric is a class of g -natural metrics and it is Riemannian if and only if $a > 0$ and $\alpha = a(a+c) - b^2 > 0$ hold. Also, for $a = 1$ and $b = c = 0$, the metric G reduces to the Sasaki lift metric (See [1]). Let $\tilde{\nabla}$ be the Levi-Civita connection of G . Then it is characterized by [1]

$$\begin{cases} (\tilde{\nabla}_{X^h} Y^h)|_t = (\nabla_X Y)^h|_t + (A(t, X, Y))^h + (B(t, X, Y))^v, \\ (\tilde{\nabla}_{X^h} Y^v)|_t = (\nabla_X Y)^v|_t + (C(t, X, Y))^h + (D(t, X, Y))^v, \\ (\tilde{\nabla}_{X^v} Y^h)|_t = (C(t, Y, X))^h + (D(t, Y, X))^v, \quad (\tilde{\nabla}_{X^v} Y^v)|_t = 0, \end{cases}$$

for all vector fields X, Y on M , where A, B, C, D are the tensor fields of type (1, 2) on M defined by

$$\begin{aligned} A(t, X, Y) &= -\frac{ab}{2\alpha}[R(X, t)Y + R(Y, t)X], \\ B(t, X, Y) &= \frac{b^2}{\alpha}R(X, t)Y - \frac{a(a+c)}{2\alpha}R(X, Y)t, \\ C(t, X, Y) &= -\frac{a^2}{2\alpha}R(Y, t)X, \quad D(t, X, Y) = \frac{ab}{2\alpha}R(Y, t)X, \end{aligned}$$

where t is thought as a vector field on M with local expression $t = y^i \frac{\partial}{\partial x^i}$. Moreover, $t^v = y^i \frac{\partial}{\partial y^i}$ is the Liouville vector field and $t^h = y^i \frac{\delta}{\delta x^i}$ is the geodesic spray of the metric g .

Theorem 1. *Let (M, g) be a Riemannian manifold and G be the Riemannian metric given by (2.3) on TM . Then the Riemannian curvature tensor \tilde{R} of (TM, G) is completely determined by*

$$\begin{aligned} \tilde{R}(X^v, Y^v)Z^v &= 0, \\ \tilde{R}(X^v, Y^v)Z^h &= \left\{ \frac{a^2}{\alpha}R(X, Y)Z + \frac{a^2}{4\alpha^2}[R(X, t)R(Y, t)Z - R(Y, t)R(X, t)Z] \right\}^h \\ &\quad + \left\{ \frac{ab}{\alpha}R(Y, X)Z + \frac{a^3b}{4\alpha^2}[R(Y, t)R(X, t)Z - R(X, t)R(Y, t)Z] \right\}^v, \\ \tilde{R}(X^h, Y^v)Z^v &= \left\{ \frac{a^2}{2\alpha}R(Z, Y)X - \frac{a^4}{4\alpha^2}R(Y, t)R(Z, t)X \right\}^h \\ &\quad + \left\{ \frac{a^3b}{4\alpha^2}R(Y, t)R(Z, t)X - \frac{ab}{2\alpha}R(Z, Y)X \right\}^v, \end{aligned}$$

$$\begin{aligned}
\tilde{R}(X^h, Y^h)Z^v &= \left\{ \frac{a^2}{2\alpha} [(\nabla_Y R)(Z, t)X - (\nabla_X R)(Z, t)Y] \right. \\
&\quad + \frac{a^3 b}{4\alpha^2} [R(X, t)R(Z, t)Y - R(Y, t)R(Z, t)X] \Big\}^h \\
&\quad + \left\{ R(X, Y)Z + \frac{ab}{2\alpha} [(\nabla_X R)(Z, t)Y - (\nabla_Y R)(Z, t)X] \right. \\
&\quad + \frac{a^2}{4\alpha} [R(X, R(Z, t)Y)t - R(Y, R(Z, t)X)t] \\
&\quad \left. + \frac{a^2 b^2}{4\alpha^2} [R(Y, t)R(Z, t)X - R(X, t)R(Z, t)Y] \right\}^v,
\end{aligned}$$

$$\begin{aligned}
\tilde{R}(X^h, Y^h)Z^h &= \left\{ R(X, Y)Z + \frac{ab}{2\alpha} [2(\nabla_t R)(X, Y)Z - (\nabla_Z R)(X, Y)t] \right. \\
&\quad + \frac{a^2}{4\alpha} [R(R(Y, Z)t, t)X - R(R(X, Z)t, t)Y] \\
&\quad + \frac{a^2 b^2}{4\alpha^2} [R(X, t)R(Y, t)Z + R(X, t)R(Z, t)Y \\
&\quad - R(Y, t)R(X, t)Z - R(Y, t)R(Z, t)X] - \frac{a^2}{2\alpha} R(R(X, Y)t, t)Z \Big\}^h \\
&\quad + \left\{ -\frac{b^2}{\alpha} (\nabla_t R)(X, Y)Z + \frac{a(a+c)}{2\alpha} (\nabla_Z R)(X, Y)t \right. \\
&\quad + \frac{ab^3}{2\alpha^2} [R(R(Y, t)Z, X)t - R(X, t)R(Z, t)Y \\
&\quad - R(R(X, t)Z, Y)t + R(Y, t)R(Z, t)X] \\
&\quad + \frac{a^2 b(a+c)}{4\alpha^2} [R(X, R(Y, t)Z)t + R(X, R(Z, t)Y)t \\
&\quad - R(Y, R(X, t)Z)t - R(Y, R(Z, t)X)t \\
&\quad - R(R(Y, Z)t, t)X + R(R(X, Z)t, t)Y] \\
&\quad \left. + \frac{ab}{2\alpha} R(R(X, Y)t, t)Z \right\}^v,
\end{aligned}$$

$$\begin{aligned}
\tilde{R}(X^h, Y^v)Z^h &= \left\{ -\frac{a^2}{2\alpha} (\nabla_X R)(Y, t)Z + \frac{a^3 b}{4\alpha^2} [R(X, t)R(Y, t)Z \right. \\
&\quad - R(Y, t)R(Z, t)X - R(Y, t)R(X, t)Z] + \frac{ab}{2\alpha} [R(X, Y)Z \\
&\quad + R(Z, Y)X] \Big\}^h + \left\{ \frac{ab}{2\alpha} (\nabla_X R)(Y, t)Z - \frac{a^2 b^2}{4\alpha^2} [R(X, t)R(Y, t)Z \right. \\
&\quad - R(Y, t)R(Z, t)X - R(Y, t)R(X, t)Z] + \frac{a^2}{4\alpha} R(X, R(Y, t)Z)t \\
&\quad \left. - \frac{b^2}{\alpha} R(X, Y)Z + \frac{a(a+c)}{2\alpha} R(X, Z)Y \right\}^v.
\end{aligned}$$

Proof. The proof is an special case of the proof of Proposition 2.9 of [1]. \square

3 Weakly symmetric Riemannian manifold (TM, G)

Let (M, g) be a Riemannian manifold. If there exist 1-forms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and a vector field A on M such that

$$(\nabla_W R)(X, Y, Z) = \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_3(Y)R(X, W)Z \\ + \alpha_4(Z)R(X, Y)W + g(R(X, Y)Z, W)A,$$

then (M, g) is called weakly symmetric. In [4], the authors proved that the relations $\alpha_2 = \alpha_3 = \alpha_4$ and $A_2 = (\alpha_2)^\sharp$ are necessary conditions to weakly symmetry of g . Thus a weakly symmetric manifold (M, g) is characterized by:

$$(\nabla_W R)(X, Y, Z) = \alpha_1(W)R(X, Y)Z + \alpha_2(X)R(W, Y)Z + \alpha_2(Y)R(X, W)Z \\ + \alpha_2(Z)R(X, Y)W + g(R(X, Y)Z, W)(\alpha_2)^\sharp. \quad (3.4)$$

Theorem 2. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle with Riemannian metric G given by (2.3). Then (TM, G) is weakly symmetric if and only if (M, g) is flat. Hence (TM, G) is flat.*

Proof. If $R = 0$, then from Theorem 1, we conclude that $\tilde{R} = 0$ and so we have (3.4). Now let (TM, G) be a weakly symmetric manifold. Then we have (3.4) for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ and \tilde{W} on TM . If we suppose $\tilde{X} = X^h, \tilde{Y} = Y^v, \tilde{Z} = Z^v$ and $\tilde{W} = W^h$, then the right side of (3.4) has the following vertical part

$$\begin{aligned} & v\{\alpha_1(W^h)\tilde{R}(X^h, Y^v)Z^v + \alpha_2(X^h)\tilde{R}(W^h, Y^v)Z^v \\ & + \alpha_2(Y^v)\tilde{R}(X^h, W^h)Z^v + \alpha_2(Z^v)R(X^h, Y^v)W^h \\ & + G(R(X^h, Y^v)Z^v, W^h)(\alpha_2)^\sharp\} = \{\alpha_1(W^h)[\frac{a^3b}{4\alpha^2}R(Y, t)R(Z, t)X \\ & - \frac{ab}{2\alpha}R(Z, Y)X] + \alpha_2(X^h)[\frac{a^3b}{4\alpha^2}R(Y, t)R(Z, t)W - \frac{ab}{2\alpha}R(Z, Y)W] \\ & + \alpha_2(Y^v)\{R(X, W)Z + \frac{ab}{2\alpha}[(\nabla_X R)(Z, t)W - (\nabla_W R)(Z, t)X] \\ & + \frac{a^2}{4\alpha}[R(X, R(Z, t)W)t - R(W, R(Z, t)X)t] + \frac{a^2b^2}{4\alpha^2}[R(W, t)R(Z, t)X \\ & - R(X, t)R(Z, t)W]\} + \alpha_2(Z^v)\{\frac{ab}{2\alpha}(\nabla_X R)(Y, t)W - \frac{b^2}{\alpha}R(X, Y)W \\ & - \frac{a^2b^2}{4\alpha^2}[R(X, t)R(Y, t)W - R(Y, t)R(W, t)X - R(Y, t)R(X, t)W] \\ & + \frac{a^2}{4\alpha}R(X, R(Y, t)W)t + \frac{a(a+c)}{2\alpha}R(X, W)Y\} \\ & + (a+c)[-\frac{a^4}{4\alpha^2}g(R(Y, t)R(Z, t)X, W) + \frac{a^2}{2\alpha}g(R(Z, Y)X, W)]\alpha_2^\sharp \\ & + b[\frac{a^3b}{4\alpha^2}g(R(Y, u)R(Z, u)X, W) - \frac{ab}{2\alpha}g(R(Z, Y)X, W)]\alpha_2^\sharp\}^v. \end{aligned} \quad (3.5)$$

Now, we compute the vertical part of the left side of (3.4). Using Theorem 1 we obtain

$$\begin{aligned}
v(\tilde{\nabla}_{W^h} \tilde{R}(X^h, Y^v) Z^v) &= \left\{ \frac{a^3 b}{4\alpha^2} \nabla_W (R(Y, t) R(Z, t) X) - \frac{ab}{2\alpha} \nabla_W R(Z, Y) X \right. \\
&\quad + \frac{a^5(a+c)}{8\alpha^3} R(W, R(Y, t) R(Z, t) X) t + \frac{a^3(a+c)}{4\alpha^2} R(W, R(Z, Y) X) t \\
&\quad - \frac{a^4 b^2}{4\alpha^3} R(W, t) R(Y, t) R(Z, t) X - \frac{a^2 b^2}{2\alpha^2} R(W, t) R(Z, Y) X \\
&\quad \left. + \frac{a^4 b^2}{8\alpha^3} R(R(Y, t) R(Z, t) X, t) W - \frac{a^2 b^2}{4\alpha^2} R(R(Z, Y) X, t) W \right\}^v, \quad (3.6)
\end{aligned}$$

$$\begin{aligned}
v(\tilde{R}(\tilde{\nabla}_{W^h} X^h, Y^v) Z^v) &= \left\{ \frac{a^3 b}{4\alpha^2} R(Y, t) R(Z, t) \nabla_W X - \frac{ab}{2\alpha} R(Z, Y) \nabla_W X \right. \\
&\quad \left. + \frac{a^3}{4\alpha^2} R(Y, t) R(Z, t) A(t, W, X) - \frac{ab}{2\alpha} R(Z, Y) A(t, W, X) \right\}^v, \quad (3.7)
\end{aligned}$$

$$\begin{aligned}
v(\tilde{R}(X^h, \tilde{\nabla}_{W^h} Y^v) Z^v) &= \left\{ \frac{ab}{2\alpha} [(\nabla_X R)(Z, t) C(t, W, Y) - (\nabla_{C(t, W, Y)} R)(Z, t) X] \right. \\
&\quad + \frac{a^3 b}{4\alpha^2} R(\nabla_W Y, t) R(Z, t) X - \frac{ab}{2\alpha} R(Z, \nabla_W Y) X + R(X, C(t, W, Y)) X \\
&\quad + \frac{a^2 b^2}{4\alpha^2} [R(C(t, W, Y), t) R(Z, t) X - R(X, t) R(Z, t) C(t, W, Y)] \\
&\quad + \frac{a^2}{4\alpha} [R(X, R(Z, t) C(t, W, Y)) t - R(C(t, W, Y), R(Z, t) X) t] \\
&\quad \left. + \frac{a^3 b}{4\alpha^2} R(D(t, W, Y), t) R(Z, t) X - \frac{ab}{2\alpha} R(Z, D(t, W, Y)) X \right\}^v, \quad (3.8)
\end{aligned}$$

$$\begin{aligned}
v(\tilde{R}(X^h, Y^v) \tilde{\nabla}_{W^h} Z^v) &= \left\{ \frac{a^3 b}{4\alpha^2} R(Y, t) R(\nabla_W Z, t) X - \frac{ab}{2\alpha} R(\nabla_W Z, Y) X \right. \\
&\quad + \frac{a^3 b}{4\alpha^2} R(Y, t) R(D(t, W, Z), t) X + \frac{ab}{2\alpha} (\nabla_X R)(Y, t) C(t, W, Z) \\
&\quad - \frac{ab}{2\alpha} R(D(t, W, Z), Y) X + \frac{a^2}{4\alpha} R(X, R(Y, t) C(t, W, Z)) t \\
&\quad - \frac{b^2}{\alpha} R(X, Y) C(t, W, Z) + \frac{a(a+c)}{2\alpha} R(X, C(t, W, Z)) Y \\
&\quad - \frac{a^2 b^2}{4\alpha^2} [R(X, t) R(Y, t) C(t, W, Z) - R(Y, t) R(C(t, W, Z), t) X \\
&\quad \left. - R(Y, t) R(X, t) C(t, W, Z)] \right\}^v. \quad (3.9)
\end{aligned}$$

Using (3.6)-(3.9) we have $v((\tilde{\nabla}_{W^h} \tilde{R})(X^H, Y^V) Z^V)$. Now we consider the following

$$v((\tilde{\nabla}_{W^H} \tilde{R})(X^H, Y^V) Z^V) = (3.5). \quad (3.10)$$

Setting $Y = t$ in the above equation implies

$$\begin{aligned}
& -\frac{ab}{2\alpha}\alpha_1(W^h)R(Z, t)X - \frac{ab}{2\alpha}\alpha_2(X^h)R(Z, t)W + \alpha_2(t^v)\{R(X, W)Z \\
& + \frac{ab}{2\alpha}[(\nabla_X R)(Z, t)W - (\nabla_W R)(Z, t)X] + \frac{a^2}{4\alpha}[R(X, R(Z, t)W)t \\
& - R(W, R(Z, t)X)t] + \frac{a^2b^2}{4\alpha^2}[R(W, t)R(Z, t)X - R(X, t)R(Z, t)W]\} \\
& + \alpha_2(Z^v)[\frac{a(a+c)}{2\alpha}R(X, W)t - \frac{b^2}{\alpha}R(X, t)W] + \frac{a}{2}g(R(Z, t)X, W)\alpha_2^\# \\
& = \frac{a^2b^2}{2\alpha^2}R(W, t)R(Z, t)X - \frac{a^3(a+c)}{4\alpha^2}R(W, R(Z, t)X)t \\
& - \frac{ab}{2\alpha}(\nabla_W R)(Z, t)X - \frac{a^2b^2}{4\alpha^2}R(R(Z, t)X, t)W + \frac{ab}{2\alpha}R(Z, t)A(t, W, X) \\
& + \frac{ab}{2\alpha}R(D(t, W, Z), t)X - \frac{a(a+c)}{2\alpha}R(X, C(t, W, Z))t \\
& + \frac{b^2}{\alpha}R(X, t)C(t, W, Z). \tag{3.11}
\end{aligned}$$

Similarly, setting $Z = t$ in (3.10) gives us

$$\begin{aligned}
& -\frac{ab}{2\alpha}\alpha_1(W^h)R(u, Y)X - \frac{ab}{2\alpha}\alpha_2(X^h)R(t, Y)W + \alpha_2(Y^v)R(X, W)t \\
& + \alpha_2(t^v)\{\frac{ab}{2\alpha}(\nabla_X R)(Y, t)W - \frac{a^2b^2}{4\alpha^2}[R(X, t)R(Y, t)W - R(Y, t)R(W, t)X \\
& - R(Y, t)R(X, t)W] + \frac{a^2}{4\alpha}R(X, R(Y, t)W)t + \frac{a(a+c)}{2\alpha}R(X, W)Y \\
& - \frac{b^2}{\alpha}R(X, Y)W\} + \frac{a}{2}g(R(t, Y)X, W)\alpha_2^\# \\
& = \frac{a^2b^2}{2\alpha^2}R(W, t)R(t, Y)X - \frac{a^3(a+c)}{4\alpha^2}R(W, R(t, Y)X)t \\
& - \frac{ab}{2\alpha}(\nabla_W R)(t, Y)X - \frac{a^2b^2}{4\alpha^2}R(R(t, Y)X, t)W + \frac{ab}{2\alpha}R(t, Y)A(t, W, X) \\
& + \frac{ab}{2\alpha}R(t, D(t, W, Y))X - R(X, C(t, W, Y))t. \tag{3.12}
\end{aligned}$$

Setting $Y = Z$ in the above equation and then summing it with (3.11) derive

that

$$\begin{aligned}
& \alpha_2(Z^v) \left\{ -\frac{b^2}{\alpha} R(X, t)W + \frac{a(a+c) + 2\alpha}{2\alpha} R(X, W)t \right\} \\
& + \alpha_2(t^v) \left\{ R(X, W)Z + \frac{ab}{2\alpha} [2(\nabla_X R)(Z, t)W - (\nabla_W R)(Z, t)X] \right. \\
& + \frac{a^2}{4\alpha} [2R(X, R(Z, t)W)t - R(W, R(Z, t)X)t] \\
& + \frac{a^2 b^2}{4\alpha^2} [R(W, t)R(Z, t)X - 2R(X, t)R(Z, t)W + R(Z, t)R(W, t)X \\
& + R(Z, t)R(X, t)W] - \frac{b^2}{\alpha} R(X, Z)W + \frac{a(a+c)}{2\alpha} R(X, W)Z \left. \right\} \\
& = \frac{b^2}{\alpha} R(X, t)C(t, W, Z) - \frac{a(a+c) + 2\alpha}{2\alpha} R(X, C(t, W, Z))t. \tag{3.13}
\end{aligned}$$

Putting $Z = t$ in the above equation we get

$$\alpha_2(t^v) \left[\frac{b^2}{\alpha} R(t, X)W + \frac{a(a+c) + 2\alpha}{2\alpha} R(X, W)t \right] = 0. \tag{3.14}$$

Interchanging X and W in the above equation yields

$$\alpha_2(t^v) \left\{ \frac{b^2}{\alpha} R(t, W)X + \frac{a(a+c) + 2\alpha}{2\alpha} R(W, X)t \right\} = 0.$$

By subtracting the above equation from (3.14) we get

$$\alpha_2(t^v) \left\{ \frac{b^2}{\alpha} [R(t, X)W + R(W, t)X] + \frac{a(a+c) + 2\alpha}{\alpha} R(X, W)t \right\} = 0.$$

Using Bianchi identity in above relation gives us

$$\alpha_2(t^v) R(X, W)t = 0.$$

If $\alpha_2(t^v) \neq 0$ we have the conclusion. Now let $\alpha_2(t^v) = 0$, then from (3.13) we have

$$\begin{aligned}
& \alpha_2(Z^v) \left\{ \frac{a(a+c) + 2\alpha}{2\alpha} R(X, W)t - \frac{b^2}{\alpha} R(X, t)W \right\} \\
& = -\frac{a(a+c) + 2\alpha}{2\alpha} R(X, C(t, W, Z))t + \frac{b^2}{\alpha} R(X, t)C(t, W, Z). \tag{3.15}
\end{aligned}$$

Exchanging X and W in the above equation we obtain

$$\begin{aligned}
& \alpha_2(Z^v) \left\{ \frac{a(a+c) + 2\alpha}{2\alpha} R(W, X)t - \frac{b^2}{\alpha} R(W, t)X \right\} \\
& = -\frac{a(a+c) + 2\alpha}{2\alpha} R(W, C(t, X, Z))t + \frac{b^2}{\alpha} R(W, t)C(t, X, Z).
\end{aligned}$$

By subtracting the above equation from (3 .15) we get

$$\begin{aligned}
& \alpha_2(Z^v)\left\{\frac{a(a+c)+2\alpha}{\alpha}R(X,W)t + \frac{b^2}{\alpha}[R(W,t)X - R(X,t)W]\right\} \\
&= \frac{a(a+c)+2\alpha}{2\alpha}[R(W,C(t,X,Z))t - R(X,C(t,W,Z))t] \\
&+ \frac{b^2}{\alpha}[R(X,t)C(t,W,Z) - R(W,t)C(t,X,Z)].
\end{aligned} \tag{3 .16}$$

Using Bianchi identity in the above relation we conclude

$$\begin{aligned}
3\alpha_2(Z^v)R(X,W)t &= \frac{a(a+c)+2\alpha}{2\alpha}[R(W,C(t,X,Z))t \\
&- R(X,C(t,W,Z))t] + \frac{b^2}{\alpha}[R(X,t)C(t,W,Z) \\
&- R(W,t)C(t,X,Z)].
\end{aligned} \tag{3 .17}$$

Now, we take the g -product with t

$$\begin{aligned}
0 &= \frac{a(a+c)+2\alpha}{2\alpha}[g(R(W,C(t,X,Z))t,t) - g(R(X,C(t,W,Z))t,t)] \\
&+ \frac{b^2}{\alpha}[g(R(X,t)C(t,W,Z),t) - g(R(W,t)C(t,X,Z),t)] \\
&= -\frac{a^2b^2}{2\alpha^2}[g(R(X,t)R(Z,t)W,t) - g(R(W,t)R(Z,t)X,t)].
\end{aligned} \tag{3 .18}$$

Setting $W = t$ and $Z = X$ in the above equation we obtain

$$0 = \frac{a^2b^2}{2\alpha^2}g(R(X,t)t, R(X,t)t)$$

If $b \neq 0$ then the above equation yields $R(X,t)t = 0$ which gives us $R = 0$. Now let $b = 0$. In this case we have $\alpha = a(a+c)$ and then from (3 .15) we get

$$\alpha_2(Z^v)R(X,W)t = -R(X,C(t,W,Z))t.$$

Setting $W = X$ in the above equation gives us

$$R(X,C(t,W,Z))t = 0,$$

and consequently

$$\frac{a^2}{2\alpha}R(X,R(t,Z)X)t = 0.$$

Taking the g -product with Z we have, $g(R(X,R(t,Z)X)t, Z)$ which gives us

$$R(t,Z)X = 0.$$

Thus $R = 0$, i.e. (M, g) is flat. \square

For $\alpha_2 = 0$ respectively $\alpha_1 = 2\alpha_2$ in (3.4) we get the following result

Corollary 1. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle with Riemannian metric G given by (2.3). Then (TM, G) is recurrent or pseudo-symmetric or locally symmetric if and only if (M, g) is flat. Hence (TM, G) is flat.*

Considering $a = 1$ and $b = c = 0$ in (2.3) we get the results of [2], [3] for the Sasakian lift metric g^s .

Corollary 2. *(TM, g^s) is weakly symmetric (recurrent or pseudo-symmetric or locally symmetric) Riemannian manifold if and only if the base manifold (M, g) is flat. Hence (TM, G) is flat.*

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